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Global behavior of solutions for the Gross-Pitaevskii equation

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1. Introduction

This talk is based on joint work with Stephen Gustafson and Tai-Peng Tsai (University of British Columbia). We are interested in long-time behavior of solutions for the Gross-Pitaevskii equation (GP) for $\psi(t,x):\mathbb{R}^{1+d}\to\mathbb{C}$,

$$i\psi_t+\Delta\psi=(|\psi|^2-1)\psi,\quad |\psi(t,x)|\to 1\quad (|x|\to\infty), \tag{1.1}$$

which are used to describe various phenomena such as superfluids, Bose-Einstein condensation and nonlinear optics. (GP) has naturally the Schrödinger part because it is derived through the mean-field (Hartree) approximation from the quantum many-body system. It can be put in a hydrodynamic form by the Madelung transformation $\psi=\sqrt{\rho}e^{i\omega/2}$, where ρ and $\nabla\omega$ are regarded respectively as the density and the velocity of the superfluid. Then the equation for $(\rho,\nabla\omega)$ is given by

$$\begin{aligned}&\partial_t\rho+\nabla\cdot(\rho v)=0,\\ &\rho(\partial_t+v\cdot\nabla)v+\nabla\rho^2=\nabla\cdot(\rho\nabla^2\log\rho),\end{aligned} \tag{1.2}$$

where the last term is called quantum pressure. We will observe later that the following form of the Boussinesq equations

$$\partial_t^2u-2\Delta u+\Delta^2u=\Delta u^2, \tag{1.3}$$

which models water wave, is also very similar to (GP) both in the linear and nonlinear parts.

In the two dimensional case, if the solution ψ of (GP) is sufficiently smooth, then it has to behave around each zero point $x=x_0$ as

$$\psi(x_0+re^{i\theta})\sim r^ke^{im\theta} \tag{1.4}$$

with $k\in\mathbb{N}$, $m\in\mathbb{Z}$ and $|m|\le k$, so the zeros of ψ with $m\ne 0$ can be regarded as vortices of the superfluid in view of the Madelung transform. In fact, there are stationary vortex solutions for (GP) of the form

$$\psi(t,re^{i\theta})=\varphi_m(r)e^{im\theta},\quad \varphi_m(r)\to 1\ (r\to\infty) \tag{1.5}$$

for each nonzero $m\in\mathbb{Z}$. There exists also a family of traveling wave solutions [2]

$$\psi(t,x)=v_c(x-ct),\quad v_c(x)\to 1\ (|x|\to\infty) \tag{1.6}$$

for $0<|c|<\sqrt{2}$, which has a vortex pair when c is small. Such solutions exist also in higher dimensions [5], where vortexes concentrate on a $d-1$ dimensional sphere.

Then the important question is the stability of those solutions, namely whether small perturbation can destroy those structures, or they will recover their original shapes soon after the perturbation. Since the superfluid typically has zero viscosity and zero entropy (especially when idealized in (GP)), such stability seems to be possible only by dispersion of the disturbance.

However the dispersive property of (GP) is not so trivial, even in the simplest case $\psi = 1+$"small", due to the interaction between the small part (which is expected to disperse) and the stationary solution 1. The equation for $u := \psi - 1$ is

$$iu_t + \Delta u - 2u_1 = 3u_1^2 + 2u_2^2 + |u|^2 u, \quad u = u_1 + iu_2. \tag{1.7}$$

Specifically, the term $2u_1$ changes the linear dispersion property from the free Schrödinger equation, while the quadratic terms are difficult to treat for long-time behavior in lower spatial dimensions, even for the standard nonlinear Schrödinger equation (NLS) (cf. [11, 13]). Thus we are lead to study the asymptotic behavior of u at $t \to \infty$ in view of the nonlinear scattering theory. We need the following notations to state our scattering results:

$$U = \sqrt{-\Delta/(2-\Delta)}, \quad H = \sqrt{-\Delta(2-\Delta)}, \tag{1.8}$$

which are defined by the Fourier multipliers on $\mathbb{R}^d$. To avoid technical details, the function spaces for the uniqueness are not specified in the following theorems. $W^{k,p}$ denotes the standard Sobolev space with up to k-th derivatives, and $\dot{W}^{k,p}$ denotes the homogeneous Sobolev space with k-th derivatives only.

Theorem 1.1. [8] *Let $d = 4$. Then for any small $\varphi \in W^{1,2}(\mathbb{R}^4)$, there exists a unique global solution ψ of (GP) satisfying $\psi = 1 + Uv_1 + iv_2$, $v(0,x) = \varphi$ and*

$$\|e^{itH}v(t) - \varphi_+\|_{W^{1,2}} \to 0, \tag{1.9}$$

as $t \to \infty$, for some $\varphi_+ \in W^{1,2}(\mathbb{R}^d)$. Moreover, the map $\varphi \mapsto \varphi_+$ is a local homeomorphism around 0 in $W^{1,2}(\mathbb{R}^d)$.

The above is valid for higher dimensions $d > 4$ as well with the Sobolev regularity $(d-2)/2$. Aside from the physical relevance, the four dimensional case may be interesting for further investigation towards large data, because of the scaling of the nonlinearity containing both the L^2 critical and the $W^{1,2}$ critical terms. The proof uses only the Strichartz estimate for the linear dispersive property, which naturally leads to the restriction $d \geq 4$, because (NLS) with quadratic nonlinearity has L^2 as scaling invariant space when $d = 4$. It means that when $d < 4$ the nonlinearity becomes stronger than the linear term in view of the space-time norms of the Strichartz type.

Theorem 1.2. [9] *Let $d = 3$ and $1 < p < 3/2$. Then for any $\varphi_+ \in W\ 1,2(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3)$, there exists a unique global solution ψ of (GP) satisfying $\psi = 1+Uv_1+iv_2$ and*

$$\|v(t) - e^{-itH}\varphi_+\|_{W^{1,2}} = o(t^{-1/4}). \tag{1.10}$$

Notice that here we start only with given asymptotic profile in contrast to the above theorem where we can start either from the asymptotic data or initial data. We have a similar result in the critical case $p = 3/2$ for small φ_+. $L^{3/2}$ is scale-invariant for (NLS) with quadratic nonlinearity in three dimensions. The proof uses the $L^p - L^{p'}$ decay estimate for the linear dispersion, and hence it is limited by the so-called Strauss exponent, which is exactly 2 for $d = 3$. It means that we need some stronger dispersive estimate for $d < 3$.

Theorem 1.3. [9] *Let $d = 2$, $\varphi_+ \in \mathcal{S}$ and small in $W^{2,1}$. Then there exists a unique global solution ψ of (GP) satisfying $\psi = 1+Uv_1+iv_2$, $\|v+H^{-1}|u|^2-v_+-w\|_{H^1} \to 0$ as $t \to \infty$, where*

$$v_+ = e^{-iHt}\varphi_+, \quad w = -i\int_\infty^t e^{-iH(t-s)}|Uv_+(s)|^2 ds. \tag{1.11}$$

The modifier $w(t) \notin L_x^2$ in general, because

$$\lim_{|\xi|\to 0} |\xi|\mathcal{F}[e^{iHt}w(t)](\xi) = i\int_0^\infty \int_{\mathbf{R}^2} e^{i(\sqrt{2}-\nabla H(\eta)\cdot\theta)t}|\mathcal{F}\varphi(\eta)|^2 d\eta ds \tag{1.12}$$

and $1/|\xi| \notin L^2$. It is known that the traveling waves with finite energy does not belong to L_x^2 either, due to its spatial asymptotic $v(x) = O(|x|^{-1})$ [7].

Theorem 1.4. [10] *Let $d = 3$. Then for any $\varphi_+ \in \dot{W}^{1,2}$, there exists a global solution ψ of (GP) satisfying*

$$\psi = 1 + Uv_1 + iv_2, \quad \|v - e^{-iHt}\varphi_+\|_{H^1} \to 0 \ (t \to \infty). \tag{1.13}$$

Here we do not know uniqueness of ψ, because the proof is purely based on the compactness argument using the conservation of energy:

$$E(\psi) = \int_{\mathbf{R}^d} |\nabla\psi|^2 + \frac{(|\psi|^2-1)^2}{2} dx. \tag{1.14}$$

It is conjectured [3] that every small energy solutions disperse in three dimensions, because there is a lower bound of energy for the traveling waves [1]. The above theorem guarantees at least that there exist plenty of such solutions. On the other hand in two dimensions that is impossible because of the presence of arbitrarily small energy traveling waves [1].

The key in those results is the nonlinear transform of the solution

$$u \mapsto z := U^{-1}u_1 + H^{-1}|u|^2 + iu_2, \tag{1.15}$$

which can be characterized as the unique quadratic transform which removes simultaneously the quadratic and the cubic terms in the nonlinear part of energy.

$$E(\psi) = \int |\nabla z|^2 + \frac{(U|u|^2)^2}{2} dx. \tag{1.16}$$

The equation for z is given by

$$iz_t - Hz = 2u_1^2 - 4iH^{-1}\nabla\cdot(u_1\nabla u_2) + iU(|u|^2u_2), \tag{1.17}$$

which is surprisingly better than the equation for v (which is the linear part of z), because the bilinear terms have decay at the zero frequency $\xi = 0$, which kills the resonant interaction that could otherwise slow the time decay of solutions.

The point in the proof of the last theorem is that the nonlinear part of energy can be controlled only by the homogeneous norm $\|\nabla z\|_{L^2}$, provided that the solution is sufficiently dispersed in order to invert the transform $u \mapsto z$. In this respect, our equation for z is even better than the original cubic NLS, because the latter does not allow control of nonlinear energy without invoking the L^2 conservation law.

Finally we comment on the relation to the Boussinesq equation (1.3). Our equation (1.17) after the transformation is roughly of the form

$$iz_t - Hz = 2(\operatorname{Re} Uz)^2, \tag{1.18}$$

if we take only the first term on the right and the main term in the transform. On the other hand, if we set $w = U^{-1}u + i(-\Delta)^{-1}\dot{u}$ in (1.3), then we get exactly

$$iw_t - Hw = (\operatorname{Re} Uw)^2. \tag{1.19}$$

Therefore our results can be automatically transferred to the Boussinesq equation, except for the last theorem, which crucially depends on the energy conservation and fine structure of the nonlinearity in (GP).

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